Gravitational waves and cosmic magnetism; a cosmological approach

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Abstract

We present the formalism for the covariant treatment of gravitational radiation in a magnetized environment and discuss the implications of the field for gravity waves in the cosmological context. Our geometrical approach brings to the fore the tension properties of the magnetic force lines and reveals their intricate interconnection to the spatial geometry of a magnetised spacetime. We show how the generic anisotropy of the field can act as a source of gravitational wave perturbations and how, depending on the spatial curvature distortion, the magnetic tension can boost or suppress waves passing through a magnetized region.

1 Introduction

Magnetic fields appear everywhere in the universe. Planets, stars and galaxies carry fields that are large and extensive. The Milky Way has a magnetic field, coherent along the plane of the galaxy, with strength of a few μG. Magnetic fields also permeate the intracluster region of galaxy clusters, extending well beyond the core region of the cluster. In addition, reports of Faraday rotation in high redshift Lyman-α systems suggest that dynamically significant magnetic fields might have been present at high redshift condensations. In short, the more we look for magnetic fields in the universe the more ubiquitous we find them to be [1]-[3]. Gravitational waves, on the other hand, are as elusive as ever. An inevitable prediction of general relativity, gravitational waves are propagating ripples in the spacetime fabric triggered by changes in the matter distribution. Their extremely weak coupling to matter, however, means that detecting these ripples is a formidable task. Given the ubiquity of magnetic fields in the universe and their strong presence near some of the most promising candidates for detectable gravity wave signals, understanding the interaction between the two sources becomes especially interesting. Cosmology could provide the grounds for an exploratory first step, and recently there were two attempts in this direction [4, 5]. The former employed the covariant formalism to investigate the evolution of gravitational waves in a magnetised cosmological environment. The latter adopted a metric based approach to study gravity wave production by stochastic large-scale magnetic fields.[1] Here, following on the work of [4], we attempt to revisit the issue. Our geometrical approach brings to the fore the vector nature and the tension properties of magnetic fields and reveals the intricate interconnection between magnetic tension and spatial geometry. A direct

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[1]The issue of magnetically induced tensor (i.e. gravitational wave) signatures in the Cosmic Microwave Background spectrum has been raised and discussed in [6, 7] and also in [4].

implication of this coupling is that the magnetic contribution to spatial curvature deformations, along the field's own direction, is always zero.

We begin with a brief outline of the covariant formalism and then apply it to the treatment of gravitational wave perturbations in a weakly magnetized, almost-FRW, cosmological environment. Our background universe is a non-magnetized, spatially flat FRW cosmology filled with a single perfect fluid of very high conductivity. This model is perturbed by allowing for weak gravitational waves and a weak magnetic field and the impact of the field on the evolution of the waves is analysed. We introduce no a priori relation between the wave and the magnetic anisotropies, which enables us to look beyond the gravity-wave production issue. Throughout the analysis we employ covariantly defined tensors and use scalars that are invariantly constructed from these tensors. This means that we can account for the full spectrum of the directionally dependent magnetic effects, particularly those coming from the tension properties of the field lines. Confining to large scales we calculate the magnetic effects analytically both in the radiation and the dust dominated eras. We find that the presence of the field leaves the linear evolution of the waves unaffected, but modifies their magnitude. This implies that the overall magnetic impact depends entirely on the initial conditions. In the absence of gravity waves, we find that the anisotropic nature of the field leads to wave production. In general, however, the magnetic effect is particularly sensitive to the initial curvature deformation as measured along the direction of the field lines. This is where the subtle role of the magnetic tension becomes important. A negative curvature perturbation is found to boost the energy of the waves, whereas positive curvature leads to a damping effect. In either case the tension properties of the field tend to keep the spatial curvature deformation down to a minimum.

The layout of the paper is as follows. In Secs. 2 and 3 we provide a brief description of the covariant treatment of cosmological perturbations and of cosmic magnetic fields respectively. Section 4 presents the necessary formalism for the study of gravitational waves within weakly magnetized almost-FRW cosmologies. The equations for the linear evolution of the wave's energy density are given in Sec. 5, and in Sec. 6 we look into the magnetic effects on gravitational radiation during the radiation and the dust dominated eras.

We employ a metric with signature $(-+++)$, the spacetime indices take the values $0,1,2,3$ and use geometrised units with $c=1=8\pi G$. The Riemann and Ricci tensors are fixed by $2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d$ and $R_{ab} = R^c{}_{acb}$ respectively.

2 Covariant description of cosmological perturbations

The Ellis-Bruni approach to cosmological perturbations [8], follows earlier studies by Hawking [9] and Olson [10], and is based on Ehlers' work on covariant hydrodynamics [11]. For a detailed and updated presentation of the covariant formalism the reader is referred to [12]. The essence of the method is to identify a set of covariantly defined variables that describe the inhomogeneity and the anisotropy of the universe in a transparent and gauge invariant manner. The non-linear equations governing the dynamics of these variables are then obtained from the full field equations. The transparency of the covariant variables means that the physical and geometrical content of their dynamical equations is simple to extract. The non-linear formulae can be linearised about a chosen background yielding a set of equations that describe deviations from inhomogeneity and isotropy in a straightforward way.

In the covariant approach all physical and geometrical quantities are decomposed with respect to a fundamental timelike velocity field u_a. The latter is determined by the motion of the matter

in the universe and introduces a unique time plus space (1+3) threading of the spacetime, as opposed to the 3+1 ADM slicing. Every observer has an instantaneous three-dimensional rest space, which is the tangent hypersurface orthogonal to u_a. In a general cosmological model u_a is chosen so that it reduces to the preferred velocity at the FRW limit, thus guaranteeing the gauge invariance of the formalism. The metric $h_{ab} = g_{ab} + u_a u_b$ of the observer's rest space (g_{ab} is the spacetime metric) also defines the projected covariant derivative D_a according to $\mathrm{D}_a T_{b\ldots}{}^{c\cdots} = h_a{}^d h_b{}^e \ldots h^c{}_f \ldots \nabla_d T_{e\ldots}{}^{f\cdots}$. When u_a is irrotational, D_a reduces to the covariant derivative in the hypersurfaces orthogonal to the observer's world line. It is also convenient to introduce the derivative $\dot{T}_{b\ldots}{}^{c\cdots} = u^a \nabla_a T_{b\ldots}{}^{c\cdots}$ along the flow lines of u_a. With these definitions, the covariant derivative of the fundamental velocity field decopmoses as

$$\nabla_b u_a = \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\Theta h_{ab} - \dot{u}_a u_b \,. \tag{1}$$

The above defines the shear $\sigma_{ab} = \mathrm{D}_{\langle b} u_{a\rangle}$, the vorticity $\omega_{ab} = \mathrm{D}_{[b} u_{a]}$, the (volume) expansion scalar $\Theta = \mathrm{D}^a u_a$ and the 4-acceleration $\dot{u}_a = u^b \nabla_b u_a$ associated with the observer's motion.[2] Introducing the projected totally antisymmetric tensor $\varepsilon_{abc} = \eta_{abcd} u^d$, where η_{abcd} is the spacetime alternating tensor, we define the vorticity vector $\omega_a = \varepsilon_{abc}\omega^{bc}/2$. The latter is also written as $\omega_a = -\mathrm{curl} u_a/2$ with $\mathrm{curl} u_a = \varepsilon_{abc} \mathrm{D}^b u^c$. In addition we introduce the generalised curl of tensors by $\mathrm{curl} T_{ab\ldots c} = \varepsilon_{de(a} \mathrm{D}^d T_{b\ldots c)}{}^e$. The volume expansion defines an average length scale a, namely the scale factor of the universe, via $\dot{a}/a = \Theta/3$. The variables σ_{ab}, ω_a, and $\dot{u}_a$ characterise anisotropy in the local expansion and vanish identically in the FRW limit. The projected gradients of scalars, vector and tensors describe local inhomogeneity in the observer's instantaneous rest space and also vanish in exact FRW spacetimes.

An additional key decomposition is that of the matter stress-energy tensor T_{ab}. Relative to a fundamental observer,

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab} \,, \tag{2}$$

where $\mu = T_{ab} u^a u^b$ is the energy density, $p = h_{ab} T^{ab}/3$ is the isotropic pressure, $q_a = h_a{}^b T_{bc} u^c$ is the energy flux and $\pi_{ab} = T_{\langle ab\rangle}$ are the anisotropic stresses of the matter component. In the FRW limit q_a and π_{ab} vanish, as the energy-momentum tensor is always of the perfect-fluid form.

3 The magnetic field

3.1 Covariant description of cosmic magnetic fields

The covariant description of electromagnetic fields was given in [14], and the Ellis-Bruni approach was applied to magnetized cosmological perturbations in [15, 16]. Covariantly, the electromagnetic Faraday tensor (F_{ab}) decomposes into an electric (E_a) and a magnetic (H_a) field as

$$F_{ab} = u_{[a} E_{b]} + \varepsilon_{abc} H^c \,, \tag{3}$$

[2] Round brackets indicate symmetrization and square ones antisymmetrisation. Angled brackets indicate the projected, symmetric, trace-free part (PSTF) of second rank tensors (i.e. $S_{\langle ab\rangle} = [h_{(a}{}^c h_{b)}{}^d - (1/3) h^{cd} h_{ab}] S_{ab}$) and the projected part of vectors (i.e. $V_{\langle a\rangle} = h_a{}^b V_b$) [13].

implying that $E_a = F_{ab}u^a$ and $H_a = \varepsilon_{abc}F^{bc}/2$. Assuming that the universe is described by an infinitely conductive medium during most of its lifetime, we can ignore the presence of the electric field. Then Maxwell's equations reduce to one propagation equation

$$\dot{H}_{\langle a\rangle} = -\tfrac{2}{3}\Theta H_a + \sigma_{ab}H^b + \varepsilon_{abc}H^b\omega^c\,, \tag{4}$$

and three constraints

$$\begin{aligned} \varepsilon_{abc}\dot{u}^b H^c + \mathrm{curl}H_a &= J_{\langle a\rangle}\,, & (5)\\ 2\omega_a H^a &= \rho_{\mathrm{e}}\,, & (6)\\ \mathrm{D}_a H^a &= 0\,, & (7) \end{aligned}$$

where $J_{\langle a\rangle} = h_a{}^b J_b$ is the projected current density and $\rho_{\mathrm{e}} = -u_a J^a$ is the charge density [15]. The above determine the evolution of the magnetic field completely. Relative to the fundamental observer, the stress-energy tensor of the field decomposes as

$$\mathcal{T}_{ab} = \tfrac{1}{2}H^2 u_a u_b + \tfrac{1}{6}H^2 h_{ab} + \Pi_{ab}\,, \tag{8}$$

where $H^2 = H_a H^a$ and

$$\Pi_{ab} = -H_{\langle a}H_{b\rangle} = \tfrac{1}{3}H^2 h_{ab} - H_a H_b\,. \tag{9}$$

Note that in the absence of electric fields the electromagnetic Poynting vector vanishes. Thus, the magnetic field behaves as a special imperfect fluid with energy density $\mu_{\mathrm{m}} = H^2/2$, isotropic pressure $p_{\mathrm{m}} = H^2/6$ and anisotropic pressure Π_{ab}. The latter reflects the vector nature of the field and carries the tension properties of the magnetic force lines.

3.2 The magnetic tension

Magnetic fields exert an isotropic pressure in all directions and carry a tension along their lines of force, with each flux tube behaving like an infinitely elastic rubber band [17, 18]. The tension properties are encoded in the eigenvalues of Π_{ab}. Orthogonal to H_a there are two positive eigenvalues equal to 1/3 each. Thus, the magnetic pressure perpendicular to the field lines is positive, reflecting their tendency to push each other apart. In the H_a direction, however, the eigenvalue is -2/3 and the magnetic pressure is negative. The minus sign reflects the elasticity of the magnetic lines and their tendency to remain as "straight" as possible. It should be emphasized that the total magnetic pressure along the direction of the field lines is also negative and equals $p_{\mathrm{mg}} = -\mu_{\mathrm{mg}} = -H^2/2$. In other words, the false vacuum condition is satisfied along the magnetic lines of force.

The magnetic tension has also non-trivial implications for the geometry of a magnetised 3-dimensional space. Consider, for example, the non-linear Gauss-Codacci equation associated with a non-rotating magnetized spacetime. The Ricci curvature tensor of the spatial hypersurfaces decomposes irreducibly to [15]

$$\mathcal{R}_{ab} = \tfrac{1}{3}\mathcal{R}h_{ab} + \tfrac{1}{2}\Pi_{ab} - \tfrac{1}{3}\Theta\sigma_{ab} + \sigma_{c\langle a}\sigma^c{}_{b\rangle} + E_{ab}\,, \tag{10}$$

where

$$\mathcal{R} = 2\left(\mu + \tfrac{1}{2}H^2\right) - \tfrac{2}{3}\Theta^2 + 2\sigma^2\,, \tag{11}$$

is the associated Ricci scalar and we have assumed a infinitely conductive perfect fluid for simplicity. Ignoring all sources but the magnetic field and then contracting twice along the field lines we obtain

$$\Re \equiv \mathcal{R}_{ab}\eta^a\eta^b = \tfrac{1}{3}H^2 + \tfrac{1}{2}\Pi_{ab}\eta^a\eta^b = 0\,, \tag{12}$$

on using expression (9) for Π_{ab}. Note that $\eta_a = H_a/\sqrt{H^2}$ is the unit vector parallel to the magnetic force lines (i.e. $\eta_a u^a = 0$ and $\eta_a\eta^a = 1$). Hence, the tension properties of the field ensure that the spatial curvature along the magnetic direction is unaffected by the energy input of the field. This result demonstrates the generic tendency of the field lines to remain straight and it is indicative of what one might call a natural magnetic preference flat spatial geometry. It is also independent of the energy density of the field, namely of how close together or far apart the magnetic lines are. Thus, no matter how much energy one pumps into or removes from the field the magnetic contribution to $\mathcal{R}_{ab}\eta^a\eta^b$ remains zero.

It should be emphasised that, despite their directional dependence, the tension properties of the field can affect average scalars such as the volume expansion of the universe (see [19]-[21]). As it turns out, they can also affect the energy density of gravitational waves passing through a magnetised region.

4 Covariant description of magnetized gravitational waves

4.1 The background equations

Consider an unperturbed non-magnetized FRW universe filled with a single barotropic fluid of infinite conductivity. When the spatial sections are flat, the background model is described by two propagation equations

$$\dot{\mu} = -(1+w)\Theta\mu\,, \tag{13}$$
$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 + \tfrac{1}{2}\mu(1+3w)\,, \tag{14}$$

and one constraint

$$\mu = \tfrac{1}{3}\Theta^2\,, \tag{15}$$

where $w = p/\mu$. Assuming that the fluid retains the barotropic equation of state (i.e. $p = p(\mu)$ always) and remains highly conductive, we perturb the background allowing for weak gravitational waves and a weak magnetic field. The infinite conductivity of the medium means that we can disregard any large-scale electric fields. The weakness of the magnetic field allows us to treat its energy density, its isotropic pressure and also the anisotropic magnetic stresses as first order perturbations. This automatically ensures that all three of them are gauge-independent variables. Finally, we assume that the magnetic field is coherent on all scales of interest.

4.2 The linear equations

The covariant description of gravitational waves, in the absence of magnetic fields, was originally considered by Hawking [9], while more recent treatments can be found in [22, 23].[3] Covariantly,

[3] Although the magnetic field can be treated as a viscous fluid, the approach discussed in [9] is not applicable here, since we are not introducing any phenomenological relation between the shear and the anisotropic stresses.

we monitor gravity waves via the electric ($E_{ab} = E_{\langle ab\rangle}$) and magnetic ($H_{ab} = H_{\langle ab\rangle}$) components of the Weyl (or conformal curvature) tensor C_{abcd}. The latter describes the locally free gravitational field, namely tidal forces and gravity waves. For a fundamental observer the Weyl tensor decomposes as [24]

$$C_{abcd} = (g_{abqp}g_{cdsr} - \eta_{abqp}\eta_{cdsr})\, u^q u^s E^{pr} - (\eta_{abqp}g_{cdsr} + g_{abqp}\eta_{cdsr})\, u^q u^s H^{pr}\,, \tag{16}$$

where $g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$ is the de Witt supermetric and η_{abcd} is the 4-dimensional permutation tensor. It follows that

$$E_{ab} = C_{acbd}u^c u^d\,, \tag{17}$$
$$H_{ab} = \tfrac{1}{2}\varepsilon_{acd}C_{be}{}^{cd}u^e\,, \tag{18}$$

where $\varepsilon_{abc} = \eta_{abcd}u^d$ is the alternating tensor on the observer's 3-dimensional rest space. The electric part of the Weyl tensor plays the role of the tidal tensor associated with the Newtonian gravitational potential, while H_{ab} is essential for the propagation of gravitational radiation. Given that the Weyl tensor vanishes in FRW spacetimes, E_{ab} and H_{ab} provide a covariant and gauge invariant description of perturbations in the gravitational field. The electric and magnetic components of C_{abcd} also support the different polarisation states of propagating gravitational radiation and obey evolution equations remarkably similar to Maxwell's formulae [25]. In the presence of a weak magnetic field, the linear evolution of E_{ab} and H_{ab} is determined by the system

$$\dot{E}_{ab} = -\Theta E_{ab} + \tfrac{1}{2}\Theta\Pi_{ab} - \tfrac{1}{2}\mu(1+w)\sigma_{ab} + \mathrm{curl}H_{ab} \tag{19}$$
$$\dot{H}_{ab} = -\Theta H_{ab} - \mathrm{curl}E_{ab} + \tfrac{1}{2}\mathrm{curl}\Pi_{ab}\,, \tag{20}$$
$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\Pi_{ab} + \mathrm{D}_{\langle a}\dot{u}_{b\rangle}\,, \tag{21}$$
$$\dot{\Pi}_{ab} = -\tfrac{4}{3}\Theta\Pi_{ab}\,, \tag{22}$$

supplemented by the constrains

$$\mathrm{D}^b E_{ab} = \tfrac{1}{3}\mathrm{D}_a\mu + \tfrac{1}{4}\mathrm{D}_a H^2 - \tfrac{1}{2}\varepsilon_{abc}H^b\mathrm{curl}H^c\,, \tag{23}$$
$$\mathrm{D}^b H_{ab} = \mu(1+w)\omega_a\,, \tag{24}$$
$$\mathrm{D}^b\Pi_{ab} = \varepsilon_{abc}H^b\mathrm{curl}H^c - \tfrac{1}{6}\mathrm{D}_a H^2\,, \tag{25}$$
$$\mathrm{D}^b\sigma_{ab} = \tfrac{2}{3}\mathrm{D}_a\Theta + \mathrm{curl}\omega_a\,. \tag{26}$$

Finally, one should keep in mind that to linear order

$$H_{ab} = \mathrm{curl}\sigma_{ab} + \mathrm{D}_{\langle a}\dot{u}_{b\rangle}\,. \tag{27}$$

4.3 Isolating the tensor perturbations

In the absence of the magnetic field one isolates the pure tensor perturbations of E_{ab} and H_{ab}, namely the gravitational waves, by demanding that

$$\omega_a = 0 = \mathrm{D}_a\mu\,, \tag{28}$$

at all times. The above constraints set all the linear scalar and vector perturbations to zero, while ensuring that the remaining tensor fields are all transverse (i.e. $\mathrm{D}^a E_{ab} = \mathrm{D}^a H_{ab} = \mathrm{D}^b\sigma_{ab} = 0$). In the magnetic presence, however, one needs to impose two additional constrains

$$\mathrm{D}_a H^2 = 0 = \varepsilon_{abc}H^b\mathrm{curl}H^c\,. \tag{29}$$

In other words, the spatial gradients of the magnetic energy density vanish and the field is also force free. Restrictions (29) ensure that constraints (28) also hold in the presence of the magnetic field (see Eqs. (37), (41) in [26]). Together, conditions (28), (29) imply that $\mathrm{D}_a p = 0$, $\dot{u}_a = 0$ and $\mathrm{D}_a\Theta = 0$. In particular, the 3-gradients of the pressure vanish as a result of the barotropic equation of state. The vanishing of the 4-acceleration comes from the linearised momentum-density conservation law (see Eq. (87) in [15]). Finally, the linear propagation equation of gradients in the matter density guarantees that $\mathrm{D}_a\Theta = 0$ to first order (see Eqs. (90), (91) in [15]). The same sets of constrains also guarantee the transversality of all the associated tensor fields, as Eqs. (23)-(26) immediately verify.

5 Linear magnetized gravitational waves

5.1 Evolution of the wave anisotropy

Having isolated the tensor modes, we drop the acceleration terms from Eqs. (21) and (27). Moreover, the spatial flatness of the background means that the first order relation $H_{ab} = \mathrm{curl}\sigma_{ab}$ translates into $\mathrm{curl}H_{ab} = -\mathrm{D}^2\sigma_{ab}$, where $\mathrm{D}^2 = \mathrm{D}_a\mathrm{D}^a$ is the projected Laplacian operator. It follows that we can eliminate the magnetic Weyl tensor from Eq. (19) in favour of the shear and therefore reduce the total number of the equations by one. Thus, the linear evolution of the magnetized gravity waves is monitored by

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\Pi_{ab}\,, \tag{30}$$

$$\dot{E}_{ab} = -\Theta E_{ab} + \tfrac{1}{2}\Theta\Pi_{ab} - \tfrac{1}{2}\mu(1+w)\sigma_{ab} - \mathrm{D}^2\sigma_{ab}\,, \tag{31}$$

$$\dot{\Pi}_{ab} = -\tfrac{4}{3}\Theta\Pi_{ab}\,. \tag{32}$$

According to Eqs. (30) and (31), the field acts as a source of gravitational wave perturbations. Indeed, even when $\sigma_{ab} = 0 = E_{ab}$ initially, $\dot{\sigma}_{ab}\,, \dot{E}_{ab} \neq 0$ because of the magnetic presence. This is not surprising at all, given that magnetic fields are natural sources of anisotropic stresses. As we shall see later, however, the specific form of these stresses (i.e. the tension properties of the magnetic field lines) means that the field presence can also suppress gravity wave distortions.

Confining to large scales we can ignore the Laplacian term in the right-hand side of Eq. (31) and reduce the above given system to

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\Pi_{ab}\,, \tag{33}$$

$$\dot{E}_{ab} = -\Theta E_{ab} + \tfrac{1}{2}\Theta\Pi_{ab} - \tfrac{1}{2}\mu(1+w)\sigma_{ab}\,, \tag{34}$$

$$\dot{\Pi}_{ab} = -\tfrac{4}{3}\Theta\Pi_{ab}\,. \tag{35}$$

5.2 Evolution of the wave energy

Consider the magnitudes $\sigma^2 = \sigma_{ab}\sigma^{ab}/2$ and $E^2 = E_{ab}E^{ab}/2$ of the shear and the electric Weyl tensors respectively. Once the pure tensor perturbations have been isolated, these are the only scalars one can invariantly construct from σ_{ab} and E_{ab}. Moreover, σ^2 and E^2 provide a direct measure of the wave's energy density and amplitude[4] Their large-scale evolution comes

[4] The energy density of gravitational radiation is determined by the pure tensor (i.e. the traceless and transverse - TT) part $H^{\mathrm{TT}}_{\alpha\beta}$ of the metric perturbation (e.g. see [5])

$$\rho_{\mathrm{GW}} = \tfrac{1}{2}\,\frac{(H^{\mathrm{TT}}_{\alpha\beta})'(H_{\mathrm{TT}}^{\alpha\beta})'}{a^2}\,, \tag{36}$$

by contracting Eqs. (33) and (34) with σ_{ab} and E_{ab} respectively. In particular we find

$$(\sigma^2)^{\cdot} = -\tfrac{4}{3}\Theta\sigma^2 - \mathcal{X} - \tfrac{1}{2}H^2\Sigma\,, \tag{38}$$
$$(E^2)^{\cdot} = -2\Theta E^2 - \tfrac{1}{2}\mu(1+w)\mathcal{X} - \tfrac{1}{2}\Theta H^2\mathcal{E}\,, \tag{39}$$

with $\mathcal{X} = E_{ab}\sigma^{ab}$, $\Sigma = \sigma_{ab}\eta^a\eta^b$ and $\mathcal{E} = E_{ab}\eta^a\eta^b$, while η_a is the constant unitary vector in the direction of the magnetic force lines. The above system closes with the following propagation equations

$$\dot{\mathcal{X}} = -\tfrac{5}{3}\Theta\mathcal{X} - 2E^2 - \tfrac{1}{2}H^2\mathcal{E} - \mu(1+w)\sigma^2 - \tfrac{1}{2}\Theta H^2\Sigma\,, \tag{40}$$
$$\dot{\Sigma} = -\tfrac{2}{3}\Theta\Sigma - \mathcal{E} - \tfrac{1}{3}H^2\,, \tag{41}$$
$$\dot{\mathcal{E}} = -\Theta\mathcal{E} - \tfrac{1}{2}\mu(1+w)\Sigma - \tfrac{1}{3}\Theta H^2\,, \tag{42}$$

where the magnetic energy density simply redshifts with the expansion

$$(H^2)^{\cdot} = -\tfrac{4}{3}\Theta H^2\,. \tag{43}$$

According to Eqs. (30), (31) (or equivalently (33), (34)) the magnetic effects on σ_{ab} and E_{ab} propagate via $\Pi_{ab} = -H_{\langle a}H_{b\rangle}$. Since both σ_{ab} and E_{ab} are also trace-free tensors, the magnetic effects on σ^2 and E^2 propagate through the last two terms in Eqs. (38), (39), namely via the contractions $\Sigma = \sigma_{ab}\eta^a\eta^b$ and $\mathcal{E} = E_{ab}\eta^a\eta^b$. The latter describe the "squeezing" and the "stretching" of the space, along η_a, that is caused by the propagating gravitational wave. Crucially, η_a is the direction the magnetic tension acts along. Hence, after the scalar and the vector modes have been switched off, the only remaining linear magnetic effect comes from the tension properties of the field lines.

The scalars Σ and $\mathcal{E}$ are related to spatial curvature perturbations via the Gauss-Codacci equation. Contracting Eq. (10) twice along η_a and keeping up to first order terms we arrive at

$$\Re = \mathcal{E} - \tfrac{1}{3}\Theta\Sigma\,, \tag{44}$$

where $\Re$ is the total spatial curvature perturbation along the magnetic direction.[5] Recall that the total contribution of the field to $\Re$ is zero (see Sec. 3.2), which explains the absence of any magnetic terms in the above. Also, given that all scalar perturbations have been switched off and that the background geometry is flat, Eq. (44) contains no matter or expansion terms either.

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where a is the dimensionless scale factor, $\alpha, \beta = 1, 2, 3$ and the dash indicates conformal time derivatives. In a comoving frame the shear of the fluid flow is a 3-tensor (i.e. $\sigma_{00} = 0 = \sigma_{0\alpha}$). Also, the pure tensor part of the covariantly defined shear is related to the traceless ransverse component of the metric perturbation by $\sigma^{\rm TT}_{\alpha\beta} = a(H^{\rm TT}_{\alpha\beta})'$, with $\sigma_{\rm TT}^{\alpha\beta} = a^{-3}(H_{\rm TT}^{\alpha\beta})'$ (see [27, 28]). On using these relations Eq. (36) gives

$$\rho_{\rm GW} = \sigma^2_{\rm TT}\,, \tag{37}$$

with $\sigma^2_{\rm TT} = \sigma^{\rm TT}_{\alpha\beta}\sigma_{\rm TT}^{\alpha\beta}/2$.

[5] In [4] the scalars Σ and $\mathcal{E}$ were related to spatial curvature distortions by twice contracting the PSTF part of Eq. (10) along η_a. Here we exploit the fact that the magnetic contribution to 3-curvature perturbations (along the direction of the field lines) vanishes, and relate Σ and $\mathcal{E}$ to the total spatial curvature perturbation $\Re$. This choice does not alter the essence of the calculation, but allows for a more transparent discussion on the subtle role of spatial curvature perturbations.

6 Magnetic effects on gravitational waves

6.1 The radiation epoch

When radiation dominates $w = 1/3$, $\mu = 3/4t^2$ and $\Theta = 3/2t$ (see Eqs. (13), (15)). Then, according to Eq. (43), the magnetic energy density drops as

$$H^2 = H_0^2 \left(\frac{t_0}{t}\right)^2 , \tag{45}$$

relative to a comoving observer with $u_a = \delta_a^0 u_0$. On using the above result, we find that in the same epoch the energy of the waves, as they propagate through the magnetized radiation fluid, is governed by the equations

$$\begin{aligned} (\sigma^2)' &= -2t^{-1}\sigma^2 - \mathcal{X} - \tfrac{1}{2}\alpha t^{-2}\Sigma \,, && (46)\\ (E^2)' &= -3t^{-1}E^2 - \tfrac{1}{2}t^{-2}\mathcal{X} - \tfrac{3}{4}\alpha t^{-3}\mathcal{E} \,, && (47) \end{aligned}$$

which form a closed system with the set

$$\begin{aligned} \mathcal{X}' &= -\tfrac{5}{2}t^{-1}\mathcal{X} - 2E^2 - t^{-2}\sigma^2 - \tfrac{1}{2}\alpha t^{-2}\mathcal{E} - \tfrac{3}{4}\alpha t^{-3}\Sigma \,, && (48)\\ \Sigma' &= -t^{-1}\Sigma - \mathcal{E} - \tfrac{1}{3}\alpha t^{-2} \,, && (49)\\ \mathcal{E}' &= -\tfrac{3}{2}t^{-1}\mathcal{E} - \tfrac{1}{2}t^{-2}\Sigma - \tfrac{1}{2}\alpha t^{-3} \,, && (50) \end{aligned}$$

where $\alpha = H_0^2 t_0^2$ and the dash indicates derivatives with respect to coordinate time. Note that the zero suffix corresponds to the initial time t_0. The system (46)-(50) accepts the late-time (i.e. when $t \gg t_0$) solutions

$$\begin{aligned} \sigma^2 &= \tfrac{1}{9}\left[\sigma_0^2 + 4E_0^2 t_0^2 - 2\mathcal{X}_0 t_0 - 2\left(\mathcal{E}_0 - \tfrac{1}{2}\frac{\Sigma_0}{t_0} - \tfrac{1}{6}H_0^2\right)H_0^2 t_0^2\right] , && (51)\\ E^2 &= \tfrac{4}{9}\left[E_0^2 + \tfrac{1}{4}\left(\frac{\sigma_0}{t_0}\right)^2 - \tfrac{1}{2}\frac{\mathcal{X}_0}{t_0} - \tfrac{1}{2}\left(\mathcal{E}_0 - \tfrac{1}{2}\frac{\Sigma_0}{t_0} - \tfrac{1}{6}H_0^2\right)H_0^2\right]\left(\frac{t_0}{t}\right)^2 , && (52) \end{aligned}$$

for the magnitudes of σ_{ab} and E_{ab} respectively (see Appendix). Moreover, for radiation the twice contracted Gauss-Codacci equation (see (44)) gives

$$\Re_0 = \mathcal{E}_0 - \tfrac{1}{2}\frac{\Sigma_0}{t_0} , \tag{53}$$

which exactly coincides with the gravitational wave contribution to the parentheses in Eqs. (51), (52). On using the above, solutions (51), (52) transform into

$$\sigma^2 = \tfrac{1}{9}\left[\sigma_0^2 + 4E_0^2 t_0^2 - 2\mathcal{X}_0 t_0\right] - \tfrac{2}{9}\left(\Re_0 - \tfrac{1}{6}H_0^2\right)H_0^2 t_0^2 , \tag{54}$$

and

$$E^2 = \tfrac{4}{9}\left[E_0^2 + \tfrac{1}{4}\left(\frac{\sigma_0}{t_0}\right)^2 - \tfrac{1}{2}\frac{\mathcal{X}_0}{t_0}\right]\left(\frac{t_0}{t}\right)^2 - \tfrac{2}{9}\left(\Re_0 - \tfrac{1}{6}H_0^2\right)H_0^2\left(\frac{t_0}{t}\right)^2 , \tag{55}$$

respectively. The quantities in brackets describe the non-magnetized case. One can easily verify this by comparing our solutions to those of the magnetic-free studies [22, 23]

Results (54) and (55) show that the field leaves the evolution rate of σ^2 and E^2 unaffected, but modifies their magnitudes. Therefore, the overall magnetic impact depends entirely on the initial conditions and it is sensitive to the initial value of $\Re$. The latter describes perturbations in the spatial curvature, along the lines of the field, caused by the passing gravitational wave. In principle $\Re_0$ can take either positive or negative values.

6.2 The dust epoch

During dust domination $w = 0$ and the zero order equations guarantee that $\Theta = 2/t$ and $\mu = 4/3t^2$. Accordingly, the magnetic energy density evolves as

$$H^2 = H_0^2 \left(\frac{t_0}{t}\right)^{8/3} , \tag{56}$$

relative to a comoving observer. The evolution of σ^2 and E^2 is monitored by the system

$$\begin{aligned} (\sigma^2)' &= -\tfrac{8}{3}t^{-1}\sigma^2 - \mathcal{X} - \tfrac{1}{2}\beta t^{-8/3}\Sigma\,, &(57)\\ (E^2)' &= -4t^{-1}E^2 - \tfrac{2}{3}t^{-2}\mathcal{X} - \beta t^{-11/3}\mathcal{E}\,, &(58) \end{aligned}$$

supplemented by the set

$$\begin{aligned} \mathcal{X}' &= -\tfrac{10}{3}t^{-1}\mathcal{X} - 2E^2 - \tfrac{4}{3}t^{-2}\sigma^2 - \tfrac{1}{2}\beta t^{-8/3}\mathcal{E} - \beta t^{-11/3}\Sigma\,, &(59)\\ \Sigma' &= -\tfrac{4}{3}t^{-1}\Sigma - \mathcal{E} - \tfrac{1}{3}\beta t^{-8/3}\,, &(60)\\ \mathcal{E}' &= -2t^{-1}\mathcal{E} - \tfrac{2}{3}t^{-2}\Sigma - \tfrac{2}{3}\beta t^{-11/3}\,, &(61) \end{aligned}$$

with $\beta = H_0^2 t_0^{8/3}$. Similarly to the radiation case before, the above has the following late-time solution

$$\begin{aligned} \sigma^2 &= \tfrac{4}{25}\left[\sigma_0^2 + \tfrac{9}{4}E_0^2 t_0^2 - \tfrac{3}{2}\mathcal{X}_0 t_0\right]\left(\frac{t_0}{t}\right)^{2/3} - \tfrac{9}{50}\left(\mathcal{E}_0 - \tfrac{2}{3}\frac{\Sigma_0}{t_0} - \tfrac{1}{6}H_0^2\right)H_0^2 t_0^2\left(\frac{t_0}{t}\right)^{2/3}, &(62)\\ E^2 &= \tfrac{9}{25}\left[E_0^2 + \tfrac{4}{9}\left(\frac{\sigma_0}{t_0}\right)^2 - \tfrac{2}{3}\frac{\mathcal{X}}{t_0}\right]\left(\frac{t_0}{t}\right)^{8/3} - \tfrac{9}{50}\left(\mathcal{E}_0 - \tfrac{2}{3}\frac{\Sigma_0}{t_0} - \tfrac{1}{6}H_0^2\right)H_0^2\left(\frac{t_0}{t}\right)^{8/3}. &(63) \end{aligned}$$

In the dust era the twice contracted Gauss-Codacci equation (44) gives

$$\Re_0 = \mathcal{E}_0 - \tfrac{2}{3}\frac{\Sigma_0}{t_0}\,. \tag{64}$$

which again coincides with the wave contribution to the parentheses in Eqs. (62), (63). On using the above we may recast Eqs. (62) and (63) into

$$\sigma^2 = \tfrac{4}{25}\left[\sigma_0^2 + \tfrac{9}{4}E_0^2 t_0^2 - \tfrac{3}{2}\mathcal{X}_0 t_0\right]\left(\frac{t_0}{t}\right)^{2/3} - \tfrac{9}{50}\left(\Re_0 - \tfrac{1}{6}H_0^2\right)H_0^2 t_0^2\left(\frac{t_0}{t}\right)^{2/3}, \tag{65}$$

and

$$E^2 = \tfrac{9}{25}\left[E_0^2 + \tfrac{4}{9}\left(\frac{\sigma_0}{t_0}\right)^2 - \tfrac{2}{3}\frac{\mathcal{X}}{t_0}\right]\left(\frac{t_0}{t}\right)^{8/3} - \tfrac{9}{50}\left(\Re_0 - \tfrac{1}{6}H_0^2\right)H_0^2\left(\frac{t_0}{t}\right)^{8/3}, \tag{66}$$

respectively, where again the brackets describe the non-magnetized case [22, 23]. As with radiation before, the overall magnetic effect depends entirely on the initial conditions. Note the magneto-curvature terms in Eqs. (54), (55) and (65), (66). Qualitatively, this magneto-geometrical effect tends to reduce the energy of the wave when the initial curvature distortion is positive (i.e. for $\Re_0 > 0$), but increases both σ^2 and E^2 if $\Re_0 < 0$. Quantitatively, the effect gets stronger with increasing curvature distortion. We will return to this non-trivial behavior of the field later.

6.3 Production of gravitational waves

Consider the dust dominated era and assume that there are no gravitational waves originally present. In this case the initial conditions are $\sigma_0^2 = E_0^2 = \mathcal{X}_0 = \mathcal{E}_0 = \Sigma_0 = 0$. Then solutions (62) and (63) give

$$\sigma^2 = \tfrac{3}{10^2} H_0^4 t_0^2 \left(\frac{t_0}{t}\right)^{2/3} , \tag{67}$$

$$E^2 = \tfrac{3}{10^2} H_0^4 \left(\frac{t_0}{t}\right)^{8/3} , \tag{68}$$

with an analogous result for dust. Thus, the magnetic presence has led to gravity wave perturbations. Note that this magnetically induced production of gravitational waves results purely from the anisotropic nature of the field. Qualitatively, the tension properties of the magnetic field lines are of no consequence. That is to say any other anisotropic source would have also triggered gravitational wave distortions. Indeed, results (67), (68) remain unchanged when the minus sign of the magnetic terms in Eqs. (38)-(42) is replaced by a plus, namely when we switch from tension to ordinary positive pressure. Assuming the presence of a cosmological magnetic field at recombination we may use result (67) to estimate the strength of the induced gravitational wave. Adopting an upper limit of 10^{-9} G, in today's values (see [29]), we find an induced wave of the order of 10^{-70} GeV^4 also in today's values. The latter lies twelve orders of magnitude bellow the energy density of the inducing magnetic field.

6.4 Boosting and damping of gravitational waves

Let us now return to the magneto-curvature terms in Eqs. (54), (55) and (65), (66). Their presence allows for the possibility of a zero overall magnetic effect if $\Re = H^2/6$ initially, namely if the curvature distortion along the field lines "equals" the isotropic magnetic pressure. Alone, the magneto-curvature terms reduce the energy of the wave if $\Re_0 > 0$, but lead to an increase when $\Re_0 < 0$. The effect results directly from the tension properties of the magnetic force lines. In fact, if tension were replaced by ordinary pressure in Eqs. (38)-(42) the effect is reversed. This intricate behaviour can be seen as the field's reaction to spatial curvature deformations. More specifically, the non-linear Gauss-Codacci equation (10) gives

$$\Re = \mathcal{R}_{ab}\eta^a\eta^b = \tfrac{2}{3}\sigma^2 - \tfrac{1}{3}\Theta\sigma_{ab}\eta^a\eta^b + \sigma_{c\langle a}\sigma^c{}_{b\rangle}\eta^a\eta^b + E_{ab}\eta^a\eta^b , \tag{69}$$

for the total perturbation in the spatial curvature along the direction of the field lines. Note the absence of any magnetic terms in the above since the contribution of the field to $\mathcal{R}_{ab}\eta^a\eta^b$ is zero (see Sec. 3.2). Also, given that both matter and volume expansion perturbations have been switched off (see Sec. 3.3), $2\mu - \Theta^2/3 = 0$ due to the background flatness. The last three terms in the right-hand side of Eq. (69) describe the "stretching" and the "squeezing" of the space (in the direction of the field lines) caused by the passing gravitational wave and take positive or negative values. However on average, say over one oscillation period, their contribution to $\mathcal{R}_{ab}\eta^a\eta^b$ amounts to zero, leaving σ^2 as the only wave input to curvature deformations. Thus, by increasing σ^2 when the initial curvature is negative and by decreasing it when $\Re_0 > 0$, the field tends to minimise the curvature perturbation along its own direction. This magnetic reaction to curvature distortions is indicative of the tension properties of the field lines, that is of their tendency to remain as straight as possible. Analogous magneto-curvature effects have also been identified on the expansion of magnetised cosmologies (see [19]) and might be interpreted as an indication of a magnetic preference for spatial flatness [20, 21].

7 Discussion

The question of how magnetic fields interact with gravitational radiation is as timely as ever in view of the forthcoming gravity wave detection experiments and the ubiquity of magnetic fields in the universe. The lack of extensive research on the subject and the unique features of magnetic fields make the outcome of any such study difficult to foresee, while it could also probe unknown as yet aspects of fundamental physics. In the present article we have considered the impact of a large-scale magnetic field on gravity waves in the cosmological context. Our starting point was a spatially flat FRW background which was subsequently perturbed by weak gravitational waves and a weak magnetic field. We adopted a geometrical approach and employed the covariant formalism to examine the field effects on σ^2 and E^2, the scalars that directly describe the energy density of gravitational radiation. Throughout the analysis we maintained the pure tensor nature of the perturbed variables, employed scalars that were invariantly constructed from these tensors, and did not assume any a priori relation between the magnetic and the gravitational wave anisotropies. The geometrical nature of our approach brought to the fore the tension properties of magnetic fields and revealed their subtle interconnection with the spatial geometry of the magnetised spacetime. We found that the overall impact of the field depends on the initial set up and is particularly sensitive to the initial curvature deformation as measured along the direction of the field. In the absence of gravitational waves the magnetic presence led to wave production. Given the generically anisotropic nature of magnetic fields this is not a surprising result. The non-trivial effects came from the intricate coupling between the geometry and the tension properties of the field. The presence of the field was found to suppress the energy of gravitational waves when the initial curvature perturbation was positive, but led to a boost in the case of positive curvature deformation. Overall, the field seemed to react to the curvature distortion caused by the propagating wave and tried to keep it down to a minimum by modulating the wave's energy accordingly.

The complete dependence of the magnetic effects on the initial conditions is rather unfortunate, as at this stage it is not clear what are the best physically motivated initial configurations. The ambiguity stems from the fact that, to linear order, $\Re_0$ depends on Σ_0 and $\mathcal{E}_0$ (see Eqs. (53), (64)) which can take either positive or negative values. Ii is likely that a non-linear analysis would incorporate σ_0^2, namely the wave's original energy, thus favouring one initial configuration at the expense of the other. That aside, some interesting questions emerge when one is allowed to speculate on the basis of the linear results. Astrophysical magnetic fields, for example, are quite widespread and also considerably strong. Typical spiral galaxies carry extensive fields of few μG and compact stars can locally support magnetic fields as strong as $10^{16}\,G$. If the field presence were to boost gravity wave perturbations in general, then, depending on the efficiency of the mechanism of course, one might think that detecting gravitational waves should have been a rather straightforward task. If, on the other hand, magnetic fields can suppress gravity waves, the ubiquity of cosmic magnetism could prove a considerable setback for the forthcoming gravity wave detection projects.

Acknowledgements

The author wishes to thank Peter Dunsby, George Ellis and Roy Maartens for helpful discussions and comments. This work was supported by a Sida/NRF grant.

Appendix: Solutions

One obtains the solutions (51), (52) and (62), (63) after a straightforward but rather tedious calculation. Here, we guide the interested reader through the intermediate steps. To begin with, Eqs. (46), (47) and the set (48)-(50) accept the solutions

$$\sigma^2 = -\tfrac{1}{2}\mathcal{C}_1 + \mathcal{C}_2 t^{-3} - 2\mathcal{C}_3 t^{-3/2} - \tfrac{4}{3}\mathcal{C}_4 t^{-1} + \tfrac{4}{3}\mathcal{C}_5 t^{-5/2} + \tfrac{4}{3}\alpha^2 t^{-2}\,, \tag{70}$$

$$E^2 = -\tfrac{1}{2}\mathcal{C}_1 t^{-2} + \tfrac{1}{4}\mathcal{C}_2 t^{-5} + \mathcal{C}_3 t^{-7/2} + \tfrac{1}{3}\mathcal{C}_4 t^{-3} + \tfrac{1}{6}\mathcal{C}_5 t^{-9/2} + \tfrac{1}{12}\alpha^2 t^{-4}\,, \tag{71}$$

and

$$\mathcal{X} = \mathcal{C}_1 t^{-1} + \mathcal{C}_2 t^{-4} + \mathcal{C}_3 t^{-5/2} + \mathcal{C}_4 t^{-2} + \mathcal{C}_5 t^{-7/2} + \tfrac{2}{3}\alpha^2 t^{-3}\,. \tag{72}$$

$$\Sigma = \tfrac{2}{3}\alpha^{-1}\mathcal{C}_4 - \tfrac{2}{3}\alpha^{-1}\mathcal{C}_5 t^{-3/2} - \tfrac{4}{3}\alpha^{-1}t^{-1}\,, \tag{73}$$

$$\mathcal{E} = -\tfrac{2}{3}\alpha^{-1}\mathcal{C}_4 t^{-1} - \tfrac{1}{3}\alpha\mathcal{C}_5 t^{-5/2} - \tfrac{1}{3}\alpha t^{-2}\,, \tag{74}$$

respectively, where $\alpha = H_0^2 t_0^2$ and $\mathcal{C}_\imath$ (with $\imath = 1\,,\dots\,,5$) are the integration constants. Solutions (74), (73) immediately provide the expressions

$$\mathcal{C}_4 = -\tfrac{1}{2}\left(2\mathcal{E}_0 - \Sigma_0 t_0^{-1} - \tfrac{2}{3}H_0^2\right) H_0^2 t_0^3\,, \tag{75}$$

$$\mathcal{C}_5 = -\left(\mathcal{E}_0 + \Sigma_0 t_0^{-1} + \tfrac{5}{3}H_0^2\right) H_0^2 t_0^{9/2}\,. \tag{76}$$

which substituted into Eqs. (72)-(74) lead, after a lengthy but straightforward calculation, to the rest of the integration constants

$$\mathcal{C}_1 = -\tfrac{4}{9}\left[2E_0^2 t_0^2 + \tfrac{1}{2}\sigma_0^2 - \mathcal{X}_0 t_0 - \left(\mathcal{E}_0 - \tfrac{1}{2}\Sigma_0 t_0^{-1} - \tfrac{1}{6}H_0^2\right) H_0^2 t_0^2\right]\,, \tag{77}$$

$$\mathcal{C}_2 = \tfrac{4}{9}\left[E_0^2 t_0^2 + \sigma_0^2 + \mathcal{X}_0 t_0 + \left(\tfrac{5}{2}\mathcal{E}_0 + \tfrac{5}{2}\Sigma_0 t_0^{-1} + \tfrac{25}{12}H_0^2\right) H_0^2 t_0^2\right] t_0^3\,, \tag{78}$$

$$\mathcal{C}_3 = \tfrac{1}{9}\left[4E_0^2 t_0^2 - 2\sigma_0^2 + \mathcal{X}_0 t_0 + \left(4\mathcal{E}_0 - \tfrac{7}{2}\Sigma_0 t_0^{-1} - \tfrac{5}{3}H_0^2\right) H_0^2 t_0^2\right] t_0^{3/2}\,. \tag{79}$$

On using results (75)-(79), we arrive at the late-time (i.e. for $t \gg t_0$) solutions (51) and (52) respectively.

For the dust era we start from the system (57)-(61) and then proceed in a completely analogous way to obtain the late-time solutions (65) and (66). Here, we only provide the full solutions to the set (57)-(61). They respectively are

$$\sigma^2 = \tfrac{3}{4}\mathrm{C}_1\mathrm{t}^{-4} - \tfrac{1}{2}\mathrm{C}_2\mathrm{t}^{-2/3} - 3\mathrm{C}_3\mathrm{t}^{-7/3} - 2\mathrm{C}_4\mathrm{t}^{-2} + \tfrac{6}{7}\mathrm{C}_5\mathrm{t}^{-11/3} + 3\beta^2\mathrm{t}^{-10/3}\,, \tag{80}$$

$$E^2 = \tfrac{1}{3}\mathrm{C}_1\mathrm{t}^{-6} - \tfrac{1}{2}\mathrm{C}_2\mathrm{t}^{-8/3} + 2\mathrm{C}_3\mathrm{t}^{-13/3} + \mathrm{C}_4\mathrm{t}^{-4} + \tfrac{2}{7}\mathrm{C}_5\mathrm{t}^{-17/3} + \tfrac{3}{4}\beta^2\mathrm{t}^{-16/3}\,, \tag{81}$$

$$\mathcal{X} = \mathrm{C}_1\mathrm{t}^{-5} + \mathrm{C}_2\mathrm{t}^{-5/3} + \mathrm{C}_3\mathrm{t}^{-10/3} + \mathrm{C}_4\mathrm{t}^{-3} + \mathrm{C}_5\mathrm{t}^{-14/3} + 3\beta^2\mathrm{t}^{-13/3}\,, \tag{82}$$

$$\Sigma = \tfrac{2}{3}\beta^{-1}\mathrm{C}_4\mathrm{t}^{-1/3} - \tfrac{2}{7}\beta^{-1}\mathrm{C}_5\mathrm{t}^{-2} - 2\beta\mathrm{t}^{-5/3}\,, \tag{83}$$

$$\mathcal{E} = -\tfrac{2}{3}\beta^{-1}\mathrm{C}_4\mathrm{t}^{-4/3} - \tfrac{4}{21}\beta^{-1}\mathrm{C}_5\mathrm{t}^{-3} - 2\beta\mathrm{t}^{-8/3}\,, \tag{84}$$

where $\beta = H_0^2 t_0^{8/3}$ and C_i (with $i = 1\,,\dots\,,5$) are the integration constants.

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